Universality in thermal and electrical conductivity from holography

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Introduction

Recent developments in AdS/CFT allow us to relate various features of strongly coupled gauge theories at finite temperature with their bulk duals. For example

- Field theory at finite temperature $\equiv$ black brane in the bulk
- Entropy of the gauge theory$\equiv$ Area of the horizon of the black brane
- Hydrodynamics equations$\equiv$Equations describing the evolution of large wavelength perturbations of the black brane
- Dissipations in gauge theories $\equiv$ Absorptions into black holes.

Introduction

Various gauge theories with bulk duals exhibit several universal features.

- A prime example: $\eta/s = \frac{1}{4\pi}$, (KSS).
- Purpose of this talk: Thermal and electrical conductivities also show some universal features.
- In particular we shall show that

$$\sigma = \sigma_H \left(\frac{sT}{\epsilon + P} \right)^2$$

where σ is the conductivity of the dual gauge theory, σ_H is a geometrical quantity eveluated at the horizon. s, T, ϵ and P are respectively the entropy density, temperature, energy density and pressure of the dual gauge theory.

- And also

$$\frac{\kappa_T}{\eta T}\sum_{j=1}^{m}(\mu^j)^2 = \frac{d^2}{d-2}\Big(\frac{c^{'}}{k^{'}}\Big),$$

where κ_T is the thermal conductivity of the dual gauge theory and μ is the chemical potential. $c^{'}, k^{'}$ are determined from thermodynamics.

Plan of the talk

- Electrical and thermal conductivities: What to compute from gravity side.
- Electrical Conductivity : Observation
- Explaining observation.
- Thermal conductivity to shear viscosity ratio.
- Conclusion.

Hydrodynamics in the presence of conserved current

- $$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu J_i^\mu = 0$$
- $T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}, \quad J_I^\mu = \rho_I u^\mu + \nu_I^\mu$

  $u_\mu u^\mu = -1$
- $$\tau_{\mu\nu} = \eta\left(\partial_\mu u_\nu + \partial_\nu u_\mu - ...\right) \tag{1}$$

  $$\nu_I^\mu = -\sum_{J=1}^{m} \varkappa_{IJ}\left(\partial^\mu \frac{\mu^J}{T} + u^\mu u^\lambda \partial_\lambda \frac{\mu^J}{T}\right) \tag{2}$$
- $\eta = \frac{s}{4\pi}$.
- General result for $\varkappa_{IJ}$ not known.

Eletrical and thermal conductivity

- Thermal conductivity: Response to heat gradient in the absence to electric current ($J^\alpha = 0$).
- $$T^{t\alpha} = -\frac{1}{\sum_{I,J=1}^{m} \rho_I \varkappa_{IJ}^{-1} \rho_J} (\frac{\epsilon + P}{T})^2 (\partial^\alpha T - ...) \qquad (3)$$
- $\kappa_T = \left(\frac{\epsilon+P}{T}\right)^2 \frac{1}{\sum_{I,J=1}^{m} \rho_I \varkappa_{IJ}^{-1} \rho_J}$
- For single charge: $\kappa_T = \left(\frac{\epsilon+P}{T}\right)^2 \frac{\varkappa}{\rho^2}$.
- Electrical conductivity: $\frac{\langle J_\alpha J_\alpha \rangle}{-i\omega} = \frac{\varkappa}{T} = \sigma$.

Observation

- $\sigma_H = \frac{1}{2\kappa^2 g_{\text{eff}}^2(r)} \; g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}.$
- Obeservation: For asymptotically AdS space time.

Gravity theory in $d+1$ dimension	σ_H	σ_B	$\sigma_H(\frac{sT}{\epsilon+P})^2$
R-charge black hole in $4+1$ dim.	$\frac{N_c^2 T(1+k)^2}{16\pi(1+\frac{k}{2})}$	$\frac{N_c^2 T(2+k)}{32\pi}$	$\frac{N_c^2 T(2+k)}{32\pi}$
R-charge black hole in $3+1$ dim.	$\frac{N_c^{\frac{3}{2}}}{24\sqrt{2}\pi}(1+k)^{\frac{3}{2}}$	$\frac{(3+2k)^2 N_c^{\frac{3}{2}}}{6^3\pi\sqrt{2(1+k)}}$	$\frac{(3+2k)^2 N_c^{\frac{3}{2}}}{6^3\pi\sqrt{2(1+k)}}$
R-charge black hole in $6+1$ dim.	$\frac{4N_c^3 T^3(1+k)^3}{81(1+\frac{k}{3})^3}$	$\frac{4N_c^3 T^3(1+k)}{27(3+k)}$	$\frac{4N_c^3 T^3(1+k)}{27(3+k)}$

- Reissner Nordstrom black hole in various dimension can also be checked to follow same.

Away from conformality:

- Charged $D1$ brane(David,Mahato, Thakur and Wadia: 1008.4350): It turns out to be again

$$\sigma_B = \sigma_H (\frac{sT}{\epsilon + P})^2.$$

- Charged lifsitz like black hole: $\sigma_B \neq \sigma_H \left(\frac{sT}{\epsilon+P}\right)^2$.
- Question: What is the most general background for which above results holds?

Gravity side: assumptions and equations

- $ds^2 = -g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + g_{xx}(r)\sum_{i=1}^{d-1}(dx^i)^2$ where r is the radial coordinate.
- Rotational and translational symmetry assumed.
- $g_{tt} \sim (r - r_h), g_{rr} \sim \frac{1}{r-r_h}$. g_{xx} finite at horizon.
- Above metric includes more general cases than asymptotically AdS space.
- $S = \frac{1}{2\kappa^2}\int d^{d+1}x \ \ \sqrt{-g}(R - \frac{1}{4g_{\rm eff}^2(r)}F_{\mu\nu}F^{\mu\nu} + \text{Other fields})$.
- $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = T_{\mu\nu}^{E.M.} + T_{\mu\nu}^{Matter}$.
- $\partial_\mu\left(\frac{1}{g_{\rm eff}^2}\sqrt{-g}F^{\nu\mu}\right) = 0$.

- $A_x = \phi(r)e^{-i\omega t + iqx_1}$ and metric fluctuation $h_{tx}, \ldots$.
- Set $q = 0$. Get a equation only in terms of field ϕ.
- $$\frac{d}{dr}(N(r)\frac{d}{dr}\phi(r)) + M(r)\phi(r) = 0. \quad (4)$$
- $N(r) = \sqrt{g}\frac{1}{g_{\text{eff}}^2}g^{xx}g^{rr}$
- $M(r) = (2\kappa^2)^2\rho^2\frac{g_{rr}g_{tt}}{\sqrt{g}g_{xx}} - \omega^2 N(r)\ g_{rr}g^{tt}$

Solution

- Boundary condition:
  a.$\phi = \phi^0$ at the boundary.
  b. In going bounday condition at the horizon.
- Solution: Solve $\phi(r)$ upto linear order in ω.
- $\phi(r, \omega) = \phi_1(r) + i\omega\phi_2(r) + ...$
- To show universality we need solution in terms of back ground fields.
- Using ingoing boundary condition at the horizon to show that one needs to solve $\phi_1(r)$ only.

Computing current current corelator $\langle J_x J_x \rangle$

- Action for ϕ in the bulk

$$S = \frac{1}{2\kappa^2}\int \frac{d\omega d^{d-1}q}{(2\pi)^d} dr\Big[\frac{1}{2}N(r)\frac{d}{dr}\phi(r,\omega,q)\frac{d}{dr}\phi(r,-\omega,-q) - \frac{1}{2}M(r)\phi(r,\omega,q)\phi(r,-\omega,-q)\Big]. \quad (5)$$

- Boundary action:

$$\begin{aligned} S_{bdy} &= \lim_{r\to\infty}\frac{1}{2\kappa^2}\int \frac{d\omega d^{d-1}q}{(2\pi)^d}\left(\frac{1}{2}N(r)\frac{d}{dr}\phi(r,\omega,q)\phi(r,-\omega,-q)\right) \\ &= \lim_{r\to\infty}\int \frac{d^d q}{(2\pi)^d}\phi^0(\omega,q)\mathcal{F}(\omega,q)\,\phi^0(-\omega,-q). \quad (6) \end{aligned}$$

- Retarted corelator $G = -2\mathcal{F}(\omega,q)$.

Computing current current corelator $\langle J_x J_x \rangle$: Simplification

- Conductivity:

$$\sigma = -\lim_{\omega\to 0} \frac{2\,\Im\Big(\mathcal{F}(\omega, q=0)\Big)}{\omega} = \lim_{\omega\to 0}\frac{1}{\omega}\frac{\Im\left(\frac{1}{2\kappa^2}\left(N(r)\frac{d}{dr}\phi(r)\phi(r)\right)\right)_{\lim_{r\to\infty}}}{\phi^0\phi^0}.$$

- Use Equation for ϕ:

$$\frac{d}{dr}\Im\Big[\frac{1}{2\kappa^2}\int\frac{d^dq}{(2\pi)^d}\left(\frac{1}{2}N(r)\frac{d}{dr}\phi(r,\omega,q)\phi(r,-\omega,-q)\right)\Big] = 0. \tag{7}$$

- Can be evaluated at any radial position r, evaluate it at the horizon. (Myers, Sinha, Poulos.....)

Conductivity

- In going boundary condition:

$$\lim_{r\to r_h} \frac{d}{dr}\phi(r) = -i\omega \lim_{r\to r_h} \sqrt{\frac{g_{rr}}{g_{tt}}}\phi(r) + \mathcal{O}(\omega^2) \tag{8}$$

- Only need to solve for $\phi_1(r)$
- $\sigma_B = \frac{1}{2\kappa^2}\left(\frac{1}{g_{\rm eff}^2}\ g_{xx}^{\frac{d-3}{2}}\right)_{r=r_h}\left(\frac{\phi_1(r_h)}{\phi_1(r\to\infty)}\right)^2$
- $\sigma(r) = \frac{1}{2\kappa^2}\left(\frac{1}{g_{\rm eff}^2}\ g_{xx}^{\frac{d-3}{2}}\right)_{r=r_h}\left(\frac{\phi_1(r_h)}{\phi_1(r)}\right)^2$
- $\sigma_H = \frac{1}{2\kappa^2 g_{\rm eff}^2(r)}\ g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}.$
- $\sigma_B = \sigma_H\left(\frac{\phi_1(r_h)}{\phi_1(r\to\infty)}\right)^2$ and $\sigma(r) = \sigma_H\left(\frac{\phi_1(r_h)}{\phi_1(r)}\right)^2$

Solution: at $\mu = 0$

- $$\frac{d}{dr}(N(r)\frac{d}{dr}\phi(r)) + M(r)\phi(r) = 0. \tag{9}$$
- $\rho = 0 \Rightarrow M(r) = 0$. Only solution that is consistent solution: $\phi_1(r) = \phi^0$.
- Trivial flow from horizon to boundary (Iqbal, liu)
- Uncharged Case: We have $\sigma_B = \sigma(r) = \sigma_H = \frac{1}{2\kappa^2 g_{\text{eff}}^2(r)} \; g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h}$.

Solution: At $\mu \neq 0$.

- Since $\sigma_B = \sigma_H(\frac{sT}{\epsilon+P})^2$, and since $\sigma_B = \sigma_H \left(\frac{\phi_1(r_h)}{\phi_1(r\to\infty)} \right)^2$ take the solution of the form

$$\frac{\phi(r)}{\phi(r_h)} = 1 + \frac{\rho}{sT}(A_t(r) - A_t(r_h)).$$

- Both at r_h and boundary it produces desired result.
- Plug above solution in

$$\frac{d}{dr}(N(r)\frac{d}{dr}\phi(r)) + M(r)\phi(r) = 0.$$

- We get after littlebit of rewriting

$$\left[\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}} g_{rr}^{\frac{1}{2}}} \frac{d}{dr}(g^{xx} g_{tt})\right]_{r_h}^{r} = -2\kappa^2 \rho A_t \Big|_{r_h}^{r}.$$

- Next step: We want to relate above equation with Einstein equation.

Einstein equation

- $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = T^{E.M.}_{\mu\nu} + T^{Matter}_{\mu\nu}$.
- Evaluate $\sqrt{-g} R^t_t - \sqrt{-g} R^x_x$, and use
- $\sqrt{-g} R^t_t = -\frac{d}{dr} \left(\frac{g_{xx}^{\frac{d-1}{2}} \frac{d}{dr} g_{tt}}{2 g_{rr}^{\frac{1}{2}} g_{tt}^{\frac{1}{2}}} \right)$ and

  $\sqrt{-g} R^x_x = -\frac{d}{dr} \left(\frac{g_{xx}^{\frac{d-3}{2}} g_{tt}^{\frac{1}{2}}}{2 g_{rr}^{\frac{1}{2}}} \frac{d}{dr} g_{xx} \right)$
- $\left(\frac{g_{xx}^{\frac{d+1}{2}}}{g_{tt}^{\frac{1}{2}} g_{rr}^{\frac{1}{2}}} \frac{d}{dr} (g^{xx} g_{tt}) \right) \Bigg|_{r_h}^{r} =$

  $-2\kappa^2 \rho A_t \Bigg|_{r_h}^{r} + 2 \int_{r_h}^{r} dr \sqrt{-g} (T_t^{t,\ Matter}(r) - T_x^{x,\ Matter}(r))$
- Modulo last term, this equation is same as above.

Constraint on matter stress tensor

- Define null tangent vector $l^{\mu}\partial_{\mu} = \sqrt{-g^{tt}}\partial_t + \sqrt{g^{xx}}\partial_x$, to the constant r hyper durface.
- Can write integrand as $T^{Matter}_{\mu\nu} l^{\mu} l^{\nu} = 0$. Saturate null energy condition.
- Similar constraint was obtained by Buchel while proving universality of shear viscosity to entropy ratio. What he claimed was that $T_{\mu\nu} \sim g_{\mu\nu}(...)$, for theories obtained from supergravity. As an example consider scalr field which only depends on radial cordinate r, then $T_{\mu\nu} \sim \; g_{\mu\nu}(...)$.
- Does it explains our previous observations?

Constraint on matter stress tensor: Conductivity

- For conformal and away from conformal theory: $\sigma_B = \sigma_H \left(\frac{sT}{\epsilon+P} \right)^2$.
- For lifshitz like theories: $\sigma_B \neq \sigma_H \left(\frac{sT}{\epsilon+P} \right)^2$ since $T_t^{t,\ Matter}(r) - T_x^{x,\ Matter}(r) \neq 0$.

Conclusion: Single charge

- $$\sigma_B = \frac{1}{2\kappa^2 g_{\text{eff}}^2(r)} g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h} \frac{(sT)^2}{(\epsilon+P)^2} = \sigma_H \frac{(sT)^2}{(\epsilon+P)^2}, \qquad (10)$$

- $$\lambda = -\frac{i}{\omega}\left(\frac{g_{tt}}{g_{xx}}\right)_{r\to\infty} \frac{\rho^2}{\epsilon+P} + \frac{1}{2\kappa^2 g_{\text{eff}}^2(r)} g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h} \frac{(sT)^2}{(\epsilon+P)^2}. \qquad (11)$$

- At any radius r_c

$$\lambda = -\frac{i}{\omega}\left(\frac{g_{tt}}{g_{xx}}\right)_{r_c}\left(\frac{\rho^2}{\epsilon+P}\right)_{r\to\infty} + \frac{1}{2\kappa^2 g_{\text{eff}}^2(r)}\, g_{xx}^{\frac{d-3}{2}}\Big|_{r=r_h} \frac{(sT)^2}{(\epsilon+P)^2}\Big|_{r=r_c}. \qquad (12)$$

Conclusion: Flow from horizon to boundary and Multiple charge

- First law of thermodynamics $\epsilon + P = sT + \rho\mu$ goes over to $\epsilon + P \to sT$ as $r \to r_H$.
- $\sigma(r_c) = \sigma_H \left(\frac{sT}{\epsilon+P}\right)^2\Big|_{r_c}$.
- $\sigma(r_c = r_h) = \sigma_H \left(\frac{sT}{\epsilon+P}\right)^2\Big|_{r_c=r_h} = \sigma_H$
- $$\rho_i \sigma^{-1}_{ij} \rho_j = \rho_i \sigma^{-1}_{H,ii} \rho_i \left(\frac{\epsilon + P}{sT}\right)^2, \tag{13}$$

Thermal conductivity

- Single charge: Using $\kappa_T = \left(\frac{\epsilon+P}{\rho}\right)^2 \frac{\sigma}{T}$, we get $\kappa_T = \sigma_H T \frac{s^2}{\rho^2}$.
- Multiple charge: $\kappa_T = s^2 T \frac{1}{\rho_i \sigma^{-1}_{H,\ ii} \rho_i}$.
- Thermal conductivity to viscosity ratio:
- Singlecharge: $\frac{\kappa_T}{\eta T}\mu^2 = 8\pi^2 \frac{1}{2\kappa^2 g^2_{\mathrm{eff}}(r)} g_{xx}^{d-2}\Big|_{r=r_H} \frac{1}{(\frac{\rho}{\mu})^2}$.
- Multiplecharge:
  $\frac{\kappa_T}{\eta T}\sum_{j=1}^{m}(\mu^j)^2 = 8\pi^2 \frac{1}{2\kappa^2 \sum_{j=1}^{m} \rho_i g^2_{\mathrm{eff,ii}}(r)\rho_i} g_{xx}^{d-2}\Big|_{r=r_H} \sum_{j=1}^{m}(\mu^j)^2$.
- In the following we shall discuss universality of this ratio.

Universal thermal conductivity at non zero μ

Gravity theory in $d+1$ dimension	$\frac{\kappa_T}{\eta T}\sum_{j=1}^{m}(\mu^j)^2$	$\frac{d^2}{d-2}\left(\frac{c'}{k'}\right)$
R-charge B.H. in $4+1$ dim.	$8\pi^2$	$8\pi^2$
R-charge B.H. in $3+1$ dim.	$32\pi^2$	$32\pi^2$
R-charge B.H. in $6+1$ dim.	$2\pi^2$	$2\pi^2$
Reissner-Nordstrom B.H. in $3+1$ dim.	$4\pi^2\gamma^2$	$4\pi^2\gamma^2$
All the above cases with $\rho=0$	no change	no change

- Thermal conductivity to viscosity ratio is independent of whether charge is present or how many of them. Just like η/s.
- Question:Is there any relation between above numbers?
- To answer this lets look at zero chemical potential case first.

Thermal conductivity to viscosity ratio at zero chemical potential

- Consider CFT at $T, \mu = 0 \Rightarrow$ black hole in AdS.
- Thermodynamics

$$P = c' T^d \qquad \chi = k' T^{d-2} \tag{14}$$

$$\epsilon + P = Ts \tag{15}$$

- Transport coefficient:

$$\eta = \frac{d}{4\pi} c' T^{d-1}, \qquad \sigma = \frac{1}{d-2} \frac{d}{4\pi} k' T^{d-3} \tag{16}$$

- $\frac{\kappa_T}{\eta T}\mu^2 = \frac{d^2}{d-2}\left(\frac{c'}{k'}\right)$.

Thermal conductivity to viscosity ratio at finite chemical potential

- Consider CFT at $T, \mu \neq 0 \Rightarrow$ charged black hole in AdS.
- Thermodynamics

$$P = c^{'} T^d f_p(m) \qquad \chi = k^{'} T^{d-2} f_\chi(m) \qquad (17)$$

$$\epsilon + P = Ts + \rho\mu, \qquad m = \frac{\mu}{T} \qquad (18)$$

- Transport coefficient:

$$\eta = \frac{d}{4\pi} c^{'} T^{d-1} f_\eta(m), \qquad \sigma = \frac{1}{d-2} \frac{d}{4\pi} k^{'} T^{d-3} f_\sigma(m) \quad (19)$$

- $\frac{\kappa_T}{\eta T}\mu^2 = \frac{d^2}{d-2}\left(\frac{c^{'}}{k^{'}}\right)\left(\frac{f_p^2 f_\sigma}{f_\chi^2 f_\eta}\right)$.
- $\frac{f_p^2 f_\sigma}{f_\chi^2 f_\eta}|_{m \neq 0} = 1$.
- $\sigma = \frac{1}{d-2}\frac{d}{4\pi} k^{'} T^{d-3} \frac{f_\chi^2 f_\eta}{f_p^2}$. Fully determined by thermodynamics.

Ratio interms of central charges of dual gauge theory.

- CFT at small length scale

$$\langle J_\mu(x) J_\nu(0)\rangle \sim \frac{k}{x^{2(d-1)}}(...), \quad \langle T_{\mu\nu}(x) T_{\alpha\beta}(0)\rangle \sim \frac{c}{x^{2d}}(...), \tag{20}$$

- c, k measure total and charged degree of freedom. So we expect $k \leq c$.
- It was shown by kovtun and ritz that

$$\frac{c^{'}}{c} = \frac{1}{4\pi^{\frac{d}{2}}}\left(\frac{4\pi}{d}\right)^{d}\frac{\Gamma(d/2)^3}{\Gamma(d)}\frac{d-1}{d(d-1)}, \quad \frac{k^{'}}{k} = \frac{1}{2\pi^{\frac{d}{2}}}\left(\frac{4\pi}{d}\right)^{d-2}\frac{\Gamma(d/2)^3}{\Gamma(d)} \tag{21}$$

- $\frac{\kappa_T}{\eta T}\mu^2 = \frac{d^2}{d-2}\left(\frac{c^{'}}{k^{'}}\right) = 8\pi^2\frac{d-1}{d^3(d+1)}\frac{c}{k}$.

Conclusion

- Though I have only discussed conformal systems, it turns out that away from conformality also, similar result exist.
- $\frac{\kappa_T}{\eta T}\mu^2 = 8\pi^2 \frac{1}{2\kappa^2 g_{\text{eff}}^2(r)} g_{xx}^{d-2}\Big|_{r=r_H} \frac{1}{(\frac{\rho}{\mu})^2}.$
- Right hand side of above equation should be independent of T, μ and some universal number.
- This remains to be understood. This will require more information on $T_{\mu\nu}^{Matter}$ and gauge coupling. In particular how it depends on matter fields.